Approximation Method for Rate of Development of Temperature Distributions in Cylindrical Foods

Arthur E. Grosser*

Department of Chemistry, McGill University, 801 Sherbrooke Street West, Montreal, Quebec H3A 2K6, Canada

Approximations are found for the thermal distributions that develop in cylindrical foods heated in an infinite heat bath. By using the concept of the half-heating time, the time elapsed for the temperature at a given position in the food to increase by half of its ultimate increase, simple relations are found which may aid in estimating these distributions, the rate at which they develop, and the functionalities of their dependence on the properties of the food.

INTRODUCTION

Foods are routinely heated from ambient temperature, developing thermal gradients that drive the rate constants for the subsequent chemical transformations. Thus, an understanding of the properties (and their functionalities) that govern the rate of development of these gradients is central to food technology. Unfortunately, the heat flow equations for infinite cylinders in an infinite heat bath are of sufficient complexity that it is difficult to estimate the rates of development of these temperature distributions, even in a semiquantitative manner, or even the manner in which they depend on the properties of the food.

An approximation is presented which offers a simple method of estimating these rates and relations.

MODEL FOR CYLINDRICAL FOODS

For the purposes of this model the food object will be assumed to be an infinite cylinder of uniform composition, where the radial distance is r and the outer radius is a. It is initially at a uniform ambient temperature, T_a, and at time, t, equal to zero, is placed in an infinite heat bath at temperature T_b.

The heat flow equations for a cylinder (Carslaw and Jaeger, 1959) yield the equation for the temperature, $T(r,t)$, at the radius, r, and time, t

$$[T(r,t) - T_a]/\Delta T = 1 - 2\sum_{i=1}^{\infty} \{\exp(-\kappa\beta_i^2 t/a^2)\}[J_0(\beta_i r/a)/\beta_i J_1(\beta_i)] \quad (1)$$

where ΔT is $T_b - T_a$, κ is the thermal diffusivity, equal to K/C_p, K being the thermal conductivity and C_p the volumetric heat capacity, J_0 and J_1 are the zero-order and first-order Bessel functions, respectively, and β_i is the ith root of the zero-order Bessel functions.

HALF-HEATING TIME APPROXIMATION

A property that has been profitably used in cooling-time studies is the half-cooling time (Mohsenin, 1980), which is the time at which the temperature difference between the object and its surroundings is half the initial temperature difference. By analogy, we define the "half-heating time," $\tau_{1/2}$, to be the time at which the temperature difference between the food and its original ambient temperature is half of the initial temperature difference, $T_b - T_a$. That is, at $\tau_{1/2}$

$$T(r,\tau_{1/2}) - T_a = 0.5(T_b - T_a)$$

The half-heating condition yields

$$0.5 = 1 - 2\{\sum_{i=1}^{\infty} \exp(-\kappa\beta_i^2\tau_{1/2}/a^2)\}[J_0(\beta_i r/a)/\beta_i J_1(\beta_i)] \quad (2)$$

The first approximation is to truncate the sum after one term:

$$1/4 \cong \{\exp(-\kappa\beta_1^2\tau_{1/2}/a^2)\}[J_0(\beta_1 r/a)/\beta_1 J_1(\beta_1)] \quad (3)$$

Taking the natural logarithm of eq 3

$$\ln J_0(\beta_1 r/a) - \ln[\beta_1 J_1(\beta_1)/4] \cong \kappa\beta_1^2\tau_{1/2}/a^2 \quad (4)$$

Rearranging

$$\tau_{1/2} \cong (a^2/\kappa\beta_1^2)\{\ln J_0(\beta_1 r/a) - \ln[\beta_1 J_1(\beta_1)/4]\} \quad (5)$$

The second approximation is to use the series expression for the Bessel functions (Sokolnikoff and Sokolnikoff, 1941)

$$J_0(x) = 1 - (x^2/2^2) + x^4/2^4(2!)^2 - x^6/2^6(3!)^2 + \ldots (-1)^k x^{2k}/2^{2k}(k!)^2 \quad (6)$$

and to truncate J_0 after the second term:

$$J_0 = 1 - (x^2/4) \quad (7)$$

Now the term $\ln J_0$ can, in its turn, be expanded (Abramowitz and Stegun, 1970)

$$\ln J_0(x) = \ln[1 - (x^2/4)] = -x^2/4 + \ldots \quad (8)$$

and this series is truncated after the first term:

$$\ln J_0(x) = -x^2/4 \quad (9)$$

Now eq 5 has the form

$$\tau_{1/2} = \{a^2/[\kappa(\beta_1)^2]\}\{-x^2/4 - \ln[\beta_1 J_1(\beta_1)/4]\} \quad (10)$$

where $x = \beta_1 r/a$. Inserting the values for β_1 and $J_1(\beta_1)$, namely, 2.40483 and 0.519147, respectively (Arfken, 1985)

$$\tau_{1/2} = (a^2/\kappa)[0.20133 - (r^2/4a^2)] \quad (11)$$

This is our final equation for the half-heating time.

TESTING THE HALF-HEATING TIME EQUATION

We will compare the results of the predictions of the half-heating time eq 11 with the exact numerical results

* e-mail Internet: art_grosser@maclan.mcgill.ca

0021-8561/94/1442-0166$04.50/0

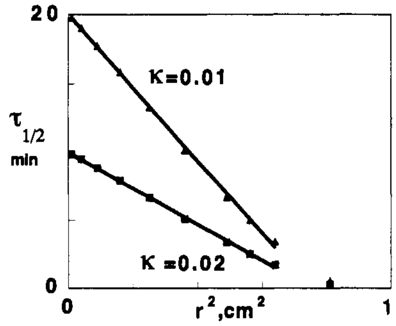

Figure 1. Half-heating times, $\tau_{1/2}$ (min), from exact computation vs $(r/a)^2$ for κ = 0.010 and 0.020 cm²/min, with a held constant at 1.0 cm. The lines are the best linear fits for $r < 0.9$ cm.

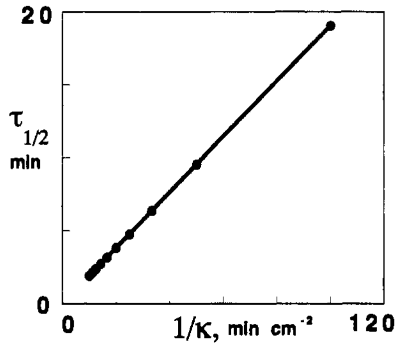

Figure 2. Half-heating times, $\tau_{1/2}$ (min), from exact computation vs $1/\kappa$ (min cm^{-2}) with r and a held constant at 0.2 and 1.0 cm, respectively. The line is the best linear fit.

that simulate the experimental behavior of the system, namely, eq 1. The computational method used to solve eq 1 numerically is to employ the series expansions for the Bessel functions (eq 6) carried out to eight terms (k = 7) for J_0 and seven terms (k = 7) for J_1, for values of the argument x less than 5. The values of these functions for larger values of x were calculated from the expressions (Sokolnikoff and Sokolnikoff, 1941)

$$J_0(x) = (2/\pi x)^{1/2}\cos[x - (\pi/4)]$$

$$J_1(x) = (2/\pi x)^{1/2}\cos[x - (\pi/4) - (\pi/2)]$$

The sum in eq 1 was carried to three terms (i = 1–3).

Heating time curves were generated for selected values of r, a, and κ and the exact half-heating times, $\tau_{1/2}$, were determined. Parameters common to all models were T_a = 293 K and T_b = 373 K. This constituted the data set against which the approximate eq 11 for $\tau_{1/2}$ was tested.

If the half-heating time approximation, eq 11, is valid, a plot of the half-heating time vs the square of r should be linear with an intercept equal to $0.20133a^2/\kappa$ and a slope equal to $-1/4\kappa$. Figure 1 displays a graph of exact half-heating times from eq 1 vs r^2 for two models over a range of r/a values from 0.1 to 0.9. The linear fits shown are for data points with r/a less than 0.9 and yield R values of 0.999. The slopes and intercepts (and their values predicted from eq 11 given in parentheses) are −25.2 (−25.0) and −12.6 (−12.5) min/cm² and 19.8 (20.1) and 9.88 (10.1) min. (If all of the data points are treated, the R values become 0.997.)

It is also possible to test the behavior of the half-heating time as a function of the other variables in eq 11, namely κ and a. Figures 2 and 3 show the exact half-heating times as functions of $1/\kappa$ and a^2, respectively, for which this equation predicts linear relations. Both plots are very linear (R = 1), with slopes and intercepts (and their

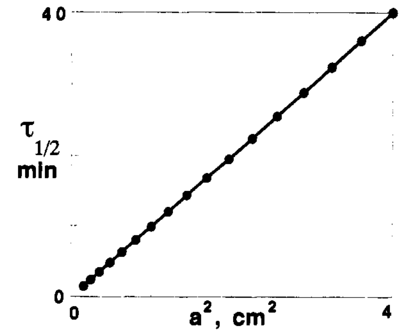

Figure 3. Half-heating times, $\tau_{1/2}$ (min), from exact computation vs a^2 (cm²), for r and κ held constant at 0.1 cm and 0.02 cm²/min, respectively. The line is the best linear fit.

predicted values from eq 11) of 0.1904 (0.1913) cm² and 8.9×10^{-4} (0) for Figure 2 and 10.03 (10.07) min/cm² and −0.126 (−0.125) min for Figure 3.

Clearly eq 11 is valid over most of the range of the variables but fails at large values of r/a, as seen in Figure 1. Is this consistent with the approximations that have been made?

The conditions that will render the equation invalid should occur when either of the two approximations, eqs 3 and 9, fail.

Condition 12 implies that the approximation should fail at small values of $\kappa\tau_{1/2}/a^2$.

$$\exp(-\kappa\beta_1^{\,2}\tau_{1/2}/a^2)xJ_0(\beta_1 r/a)/\beta_1 J_1(\beta_1)] \approx \exp(-\kappa\beta_2^{\,2}\tau_{1/2}/a^2)[xJ_0(\beta_2 r/a)\beta_2 J_1(\beta_2)] \quad (12)$$

This corresponds to the expectation that the approximation will fail close to the periphery (small $\tau_{1/2}$). It may also be thought of as a condition of the ratio $\tau_{1/2}/[a^2/\kappa]$. If a^2/κ is denoted as the "response time", t_r, then the above condition for validity becomes

$$\tau_{1/2} \gg t_r$$

Equation 9 is valid for small values of r/a, which give rise to longer half-heating times. Thus, both approximations lead to the conclusion that the approximation will be valid away from the periphery at longer heating times.

DISCUSSION

The approximate equation for $\tau_{1/2}$ shows that it depends on the radial distance, r, in a quadratic manner:

$$\tau_{1/2} = -(1/4\kappa)r^2 + 0.20133a^2/\kappa \quad (13)$$

Thus, for material of given composition and fixed κ, $\tau_{1/2}$ will change with the square of the radial distance within the cylindrical object with a sensitivity inversely proportional to the thermal diffusivity. For a given value of r, $\tau_{1/2}$ will depend only on κ and a, again in expected manners: for large κ, the half-heating times becomes small, and for large a they become large.

In general, $\tau_{1/2}$ is independent of the initial ambient temperature, T_a, the bath temperature, T_b, and even their difference, $T_b - T_a$. A quick qualitative sketch of the thermal gradients can be made if one recognizes that $\tau_{1/4}$ and $\tau_{3/4}$ are given by

$$\tau_{1/4} = (a^2/\kappa)\{0.13176 - [(r/a)^2/4]\} \quad (14)$$

$$\tau_{3/4} = (a^2/\kappa)\{0.32119 - [(r/a)^2/4]\} \quad (15)$$

In conclusion, the concept of the half-heating time offers a simple manner of estimating the rate of development of temperature distributions and the manner in which they depend on the physical properties of the system. For cylindrical objects it is approximately given by a remarkably simple equation whose accuracy extends over a wide range of conditions.

ACKNOWLEDGMENT

I thank the Vernon W. Krieble Foundation and the Department of Chemistry for their generous support.

LITERATURE CITED

Abramowitz, M.; Stegun, I. A. *Handbook of Mathematical Functions*; Dover: New York, 1970; p 68.

Arfken, G. B. *Mathematical Methods for Physicists*, 3rd ed.; Academic: New York, 1985; p 485.

Carslaw, H. S.; Jaeger, J. C. *Conduction of Heat in Solids*, 2nd ed.; Oxford; London, 1959; p 199.

Mohsenin, N. N. *Thermal Properties of Foods and Agricultural Materials*; Gordon and Breach: New York, 1980; p 13.

Sokolnikoff, I. S.; Sokolnikoff, S. S. *Higher Mathematics for Engineers and Physicists*; McGraw-Hill: New York, 1941; pp 336–337.

Received for review May 4, 1993. Accepted September 21, 1993.[⊗]

[⊗] Abstract published in *Advance ACS Abstracts*, November 15, 1993.